

# Twist Three Distribution $e(x)$ : Sum Rules and Equation of Motion Relations

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## Abstract

We investigate the twist three distribution function  $e(x)$  in light-front Hamiltonian perturbation theory. In light-front gauge, by eliminating the constrained field, we find a mass term, an intrinsic transverse momentum dependent term and a 'genuine twist three' quark-gluon interaction term in the operator. The equation of motion relation, moment relation and the sum rules are satisfied for a quark at one loop. We compare the results with other model calculations.

## Introduction

Twist three parton distribution  $e(x)$  has not been explored broadly in the literature so far.  $e(x)$  is spin independent and chiral odd. Therefore it is difficult to measure it experimentally as it can combine only with another chiral odd object. It was shown in [1] that  $e(x)$  contributes to the unpolarized Drell-Yan process at the level of twist four.  $e(x)$  enters together with the chirally odd Collin's fragmentation function  $H_1^\perp$  in the azimuthal asymmetry in semi-inclusive deep inelastic scattering (DIS) of longitudinally polarized electrons off unpolarized nucleons. This asymmetry was measured [2–4], however, apart from  $e(x)$ , several other distribution and fragmentation functions also appear in this asymmetry. In [5], another possibility of measuring  $e(x)$  is proposed through two hadron production in polarized semi-inclusive DIS; where  $e(x)$  is coupled to a two hadron fragmentation function. The first calculation of  $e(x)$  was done in [1], in MIT bag model. Further it was calculated in chiral quark soliton model [6, 7], spectator model [8] and perturbative one loop model [9]. The  $Q^2$  evolution of  $e(x)$  has been calculated in [10]. A nice review of the general properties of  $e(x)$  and the various model calculations can be found in [11]. In chiral quark soliton model, a  $\delta(x)$  singularity was observed in  $e(x)$ , which was later found also in a perturbative calculation [9]. However, no such  $\delta(x)$  term was found in the spectator model as well as in MIT bag model. It is to be noted that as  $x = 0$  cannot be experimentally achieved, no direct experimental verification of the  $\delta(x)$  contribution is possible. Another interesting aspect of  $e(x)$  is that the first moment of the flavor singlet combination of  $e(x)$  is related to the pion nucleon sigma term  $\sigma_{\pi N}$  [1]. The flavor non-singlet part of  $e(x)$  satisfies a sum rule connecting it to the hadronic mass difference between neutron and proton [6]. The first moment of  $e(x)$  obeys the sum rule

$$\int_{-1}^1 dx e(x) = \frac{1}{2M} \langle P, S | \bar{\psi}(0)\psi(0) | P, S \rangle. \quad (1)$$

The second moment of  $e(x)$  obeys another sum rule [1]:

$$\int_{-1}^1 dx x e(x) = \frac{m}{M} N_q; \quad (2)$$

where  $m$  is the mass of the quark and  $M$  is the mass of the proton.  $N_q$  is the number of quarks of a given flavor. These sum rules are not satisfied in the bag model as the QCD equation of motion is modified in the bag. They are not satisfied in the spectator model either. In QCD

equation of motion method [11], the first sum rule is saturated by the  $\delta(x)$  contribution only, whereas in chiral quark soliton model, only a part of the contribution comes from the  $\delta(x)$  term. In [9],  $e(x)$  has been calculated for a quark dressed with a gluon at one loop in QCD in light front gauge. Starting from the Feynman diagram in  $A^+ = 0$  gauge,  $k^-$  was integrated out to get to perform the calculation in light-front time-ordered method. A  $\delta(x)$  term was found which was shown to be related to the  $k^+ = 0$  modes.

The disagreement between different model calculations for  $e(x)$  makes it an interesting object for further study. In this work, we calculate  $e(x)$  for a quark at one loop in QCD using light front Hamiltonian perturbation theory in  $A^+ = 0$  gauge. Instead of using the Feynman diagrams, we expand the state in Fock space in terms of multiparton light-front wavefunctions. The advantage is that these wave functions are Lorentz boost invariant, so we can truncate the Fock space expansion to a few particle sector in a boost invariant way. The two particle light-front wave functions (LFWFs) can be calculated analytically for a quark at one loop using the light-front Hamiltonian. The partons are on-mass shell objects having non-vanishing transverse momenta. They can be called field theoretic partons. The distribution functions can be calculated at the scale  $Q^2$  using the multiparton LFWFs. To  $O(\alpha_s)$  this is an exact calculation. Another interesting aspect is the nature of ultraviolet (UV) divergence in  $e(x)$ . Being a twist three object,  $e(x)$  contains a 'bad' component of quark field. As a result, it is expected to have a different UV divergence as compared to the logarithmic divergence in twist two distributions  $f_1(x)$ . In the following, we present our calculation.

### **$e(x)$ for a quark at one loop**

The twist three distribution  $e(x)$  is given by [1] :

$$e(x) = \frac{P^+}{M} \int \frac{dy^-}{8\pi} e^{\frac{i}{2}xP^+y^-} \langle P, S | \bar{\psi}(0)\psi(y^-) | P, S \rangle. \quad (3)$$

We introduce the projection operators  $\Lambda^\pm = \frac{1}{2}\gamma_0\gamma_\pm$  and project out the light-front good and bad components of the quark field :

$$\psi^{(\pm)} = \Lambda^\pm \psi \quad (4)$$

In light-front gauge,  $A^+ = 0$ , the 'bad' component,  $\psi^{(-)}$  is constrained, and the equation of

constraint is given by [12]

$$\psi^-(y^-) = \frac{1}{i\partial^+}(i\alpha^\perp \cdot \partial^\perp + g\alpha^\perp \cdot A^\perp + \beta m)\psi^{(+)}(y^-); \quad (5)$$

where the operator  $\frac{1}{\partial^+}$  is defined as [12]

$$\frac{1}{\partial^+}f(x^-) = \frac{1}{4} \int_{-\infty}^{\infty} dy^- \epsilon(x^- - y^-) f(y^-) \quad (6)$$

The antisymmetric step function is given by

$$\epsilon(x^-) = -\frac{i}{\pi} \mathcal{P} \int \frac{d\omega}{\omega} e^{\frac{i}{2}\omega x^-} \quad (7)$$

$\mathcal{P}$  denotes the principal value. For the dynamical field The operator in Eq. (3), can be written as

$$O_e = \bar{\psi}(0)\psi(y^-) = \psi^{(-)\dagger}(0)\gamma_0\psi^{(+)}(y^-) + \psi^{(+)\dagger}(0)\gamma_0\psi^{(-)}(y^-). \quad (8)$$

Using the constraint equation for  $\psi^{(-)}$  this can be written as:

$$O_e = O_m + O_g + O_k; \quad (9)$$

where

$$O_m = m\psi^{(+)\dagger}(0) \left[ (-\frac{\overleftarrow{1}}{i\partial^+}) + (\frac{\overrightarrow{1}}{i\partial^+}) \right] \psi^{(+)}(y^-); \quad (10)$$

$$O_k = \psi^{(+)\dagger}(0) \left[ \frac{\overrightarrow{\not{\partial}^\perp}}{\partial^+} - \frac{\overleftarrow{\not{\partial}^\perp}}{\partial^+} \right] \psi^{(+)}(y^-); \quad (11)$$

$$O_g = g\psi^{(+)\dagger}(0) \left[ \not{A}_T(\frac{\overleftarrow{1}}{i\partial^+}) + (\frac{\overrightarrow{1}}{i\partial^+})\not{A}_T \right] \psi^{(+)}(y^-). \quad (12)$$

$O_m$  is the mass term,  $O_k$  is the transverse momentum dependent term and  $O_g$  is the explicit quark-gluon interaction term, also called the ‘genuine twist three’ term.

For  $\psi^+$  we use two component formalism [12]

$$\psi^+ = \begin{pmatrix} \xi \\ 0 \end{pmatrix} \quad (13)$$

The two component field  $\xi(y)$  has the Fock space expansion

$$\xi(y) = \sum_{\lambda} \chi_{\lambda} \int \frac{dk^+ d^2 k^{\perp}}{2(2\pi)^3 \sqrt{k^+}} \left[ b_{\lambda}^{\dagger}(k) e^{iky} + d_{-\lambda}(k) e^{-iky} \right]; \quad (14)$$

with

$$\chi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (15)$$

For a dressed quark state of momentum  $P$  and helicity  $\sigma$ :

$$\begin{aligned} |P, \sigma\rangle &= \phi_1 b^{\dagger}(P, \sigma) |0\rangle \\ &+ \sum_{\sigma_1, \lambda_2} \int \frac{dk_1^+ d^2 k_1^{\perp}}{\sqrt{2(2\pi)^3 k_1^+}} \int \frac{dk_2^+ d^2 k_2^{\perp}}{\sqrt{2(2\pi)^3 k_2^+}} \sqrt{2(2\pi)^3 P^+} \delta^3(P - k_1 - k_2) \\ &\phi_2(P, \sigma | k_1, \sigma_1; k_2, \lambda_2) b^{\dagger}(k_1, \sigma_1) a^{\dagger}(k_2, \lambda_2) |0\rangle. \end{aligned} \quad (16)$$

Here  $a^{\dagger}$  and  $b^{\dagger}$  are bare gluon and quark creation operators respectively and  $\phi_1$  and  $\phi_2$  are the multiparton wave functions. They are the probability amplitudes to find one bare quark and one quark plus gluon inside the dressed quark state respectively. We introduce Jacobi momenta  $x_i$ ,  $q_i^{\perp}$  such that  $\sum_i x_i = 1$  and  $\sum_i q_i^{\perp} = 0$ . They are defined as

$$x_i = \frac{k_i^+}{P^+}, \quad q_i^{\perp} = k_i^{\perp} - x_i P^{\perp}. \quad (17)$$

Also, we introduce the wave functions,

$$\psi_1 = \phi_1, \quad \psi_2(x_i, q_i^{\perp}) = \sqrt{P^+} \phi_2(k_i^+, k_i^{\perp}); \quad (18)$$

which are independent of the total transverse momentum  $P^{\perp}$  of the state and are boost invariant. The state is normalized as,

$$\langle P', \lambda' | P, \lambda \rangle = 2(2\pi)^3 P^+ \delta_{\lambda, \lambda'} \delta(P^+ - P'^+) \delta^2(P^{\perp} - P'^{\perp}). \quad (19)$$

The two particle wave function depends on the helicities of the electron and photon. Using the eigenvalue equation for the light-cone Hamiltonian, this can be written as [13],

$$\begin{aligned} \psi_{2\sigma_1, \lambda}^{\sigma}(x, q^{\perp}) &= \frac{x(1-x)}{(q^{\perp})^2 + m^2(1-x)^2} \frac{1}{\sqrt{(1-x)}} \frac{g}{\sqrt{2(2\pi)^3}} T^a \chi_{\sigma_1}^{\dagger} \left[ -2 \frac{q^{\perp}}{1-x} - \frac{\tilde{\sigma}^{\perp} \cdot q^{\perp}}{x} \tilde{\sigma}^{\perp} \right. \\ &\quad \left. + im \tilde{\sigma}^{\perp} \frac{(1-x)}{x} \right] \chi_{\sigma} \epsilon_{\lambda}^{\perp *} \psi_1. \end{aligned} \quad (20)$$

$m$  is the bare mass of the quark,  $\tilde{\sigma}_1 = \sigma_2$ ,  $\tilde{\sigma}_2 = -\sigma_1$ .  $\psi_1$  actually gives the normalization of the state [13]:

$$|\psi_1|^2 = 1 - \frac{\alpha_s}{2\pi} C_f \int_\epsilon^{1-\epsilon} dx \frac{1+x^2}{1-x} \log \frac{Q^2}{\mu^2}, \quad (21)$$

to order  $\alpha_s$ . Here  $\epsilon$  is a small cutoff on  $x$ . We have taken the cutoff on the transverse momenta to be  $Q$ . This gives the large scale of the process. The above expression is derived using Eqs (19), (16) and (20). In the above expression, we have neglected subleading finite pieces.  $\mu$  is a small scale separating hard and soft dynamics such that  $(q^\perp)^2 \geq \mu^2$ . For a dressed quark, we get,

$$x e_k(x, Q^2) = 0; \quad (22)$$

$$x e_g(x, Q^2) = \frac{m}{M} \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \left[ \frac{x}{2} \delta(1-x) - 1 + x \right]; \quad (23)$$

$$x e_m(x, Q^2) = \frac{m}{M} \left[ \delta(1-x) + \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \left\{ \frac{1+x^2}{1-x} - \delta(1-x) \int_\epsilon^{1-\epsilon} dy \frac{1+y^2}{1-y} \right\} \right]; \quad (24)$$

where  $x e_k(x, Q^2)$ ,  $x e_g(x, Q^2)$  and  $x e_m(x, Q^2)$  are contributions from  $O_k, O_g$  and  $O_m$  respectively. In the above, we have used the normalization condition Eq. (21). One can write

$$\frac{1+x^2}{1-x} - \delta(1-x) \int_\epsilon^{1-\epsilon} dy \frac{1+y^2}{1-y} = \frac{1+x^2}{1-x}_+ + \frac{3}{2} \delta(1-x). \quad (25)$$

The plus prescription is defined in the usual way, that is  $\int_0^1 dx \frac{f(x)}{1-x}_+ = \int_0^1 dx \frac{f(x)-f(1)}{1-x}$ . For a dressed quark state,  $M$  is the renormalized mass of the quark. The bare mass  $m$  of the quark is given in terms of the renormalized mass [14]:

$$m = M \left( 1 - \frac{3\alpha_s}{4\pi} C_f \log \frac{Q^2}{\mu^2} \right). \quad (26)$$

Using this, we can write

$$x e(x, Q^2) = \delta(1-x) + \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \left[ \frac{2x}{1-x}_+ + \frac{1}{2} x \delta(1-x) \right]. \quad (27)$$

In the above result, the divergence at  $x \rightarrow 1$  gets canceled by the contribution from the normalization of the state and we get the plus prescription. The above result agrees with [9].

Note that there is no  $\delta(x)$  contribution in  $xe(x)$ . However, as shown in [9],  $\delta(x)$  contribution in  $e(x)$  is related to the  $k^+ = 0$  modes (zero modes) in light-front gauge. By choosing the prescription given by Eq. (6) we are avoiding the zero modes.  $xe(x)$  is zero if we take the quark mass to be zero.

### Sum rules and equation of motion relation

$xe(x)$  can be related to the twist two unpolarized quark distribution through the equation of motion relation [15]:

$$xe(x) = x\tilde{e}(x) + \frac{m}{M}f_1(x) \quad (28)$$

Where  $f_1(x)$  is the twist two unpolarized distribution function :

$$f_1(x) = \int \frac{dy^-}{8\pi} e^{\frac{i}{2}xP^+y^-} \langle P, S | \bar{\psi}(0)\gamma^+\psi(y^-) | P, S \rangle. \quad (29)$$

Note that Eq. (28) is unaffected by the presence of the gauge link in the definition of the parton distributions. In the above relation, we have suppressed the scale dependence.  $\tilde{e}(x)$  is the genuine twist three quark-gluon interaction part of  $e(x)$ . For a dressed quark,

$$f_1(x, Q^2) = \delta(1-x) + \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \left[ \frac{1+x^2}{1-x_+} + \frac{3}{2}\delta(1-x) \right] \quad (30)$$

So the equation of motion relation is satisfied with  $x\tilde{e}(x, Q^2) = \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \left[ \frac{1}{2}x\delta(1-x) - 1+x \right]$ .

The first moment of  $e_m(x)$  and  $e_g(x)$  are given by

$$\int_{\epsilon}^1 e_m(x, Q^2) dx = 1 + \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} (-\log \epsilon - 1); \quad (31)$$

$$\int_{\epsilon}^1 e_g(x, Q^2) dx = \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} (\log \epsilon + \frac{3}{2}). \quad (32)$$

Each part has divergence as  $x \rightarrow 0$ . However, their total contribution is free of divergence :

$$\begin{aligned} \int_{\epsilon}^1 e(x, Q^2) dx &= \int_{\epsilon}^1 (e_m(x, Q^2) + e_g(x, Q^2)) dx = 1 + \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \frac{1}{2} \\ &= \frac{1}{2M} \langle P, S | \bar{\psi}(0)\psi(0) | P, S \rangle. \end{aligned} \quad (33)$$

So the sum rule in Eq. (1) is satisfied. However, our  $x$  region is limited by 0 and 1 as this is physically allowed. Note that this implies that the sum rule is not saturated by a  $\delta(x)$

contribution, unlike what was concluded in [11]. The possibility of a divergence in this relation was observed in [1]. The second moment of  $e(x)$  becomes

$$\int_0^1 x e(x, Q^2) dx = 1 - \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \frac{3}{2} = \frac{m}{M}; \quad (34)$$

with

$$\int_0^1 dx x \tilde{e}(x, Q^2) = \int_0^1 dx \left[ \frac{x}{2} \delta(1-x) - 1 + x \right] = 0; \quad (35)$$

The *rhs* of Eq. (34) vanishes in the chiral limit  $m = 0$ . Note that this result agrees with [9], the  $\delta(x)$  present there in  $e(x)$  does not contribute to this sum rule. The relation for the  $n$ -th moment of  $e(x)$  defined by  $[e]_n = \int_0^1 dx x^{n-1} e(x)$  can be written as [16]:

$$[e]_n = [\tilde{e}]_n + \frac{m}{M} [f_1]_{n-1}. \quad (36)$$

The  $n$ -th moment is calculated using the expression above :

$$\begin{aligned} [e]_n &= 1 + \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \int_0^1 dx x^{n-2} \left[ \frac{2x}{1-x_+} + \frac{1}{2} x \delta(1-x) \right] \\ &= 1 + \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \left[ -2 \sum_{j=1}^{n-1} \frac{1}{j} + \frac{1}{2} \right]; \end{aligned} \quad (37)$$

where we have used  $\int_0^1 dx x^{n-1} \frac{1}{1-x_+} = -\sum_{j=1}^{n-1} \frac{1}{j}$ . On the *rhs*,

$$\begin{aligned} [\tilde{e}]_n &= \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \int_0^1 dx x^{n-2} \left[ \frac{x}{2} \delta(1-x) - 1 + x \right] \\ &= \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \left[ \frac{1}{2} - \frac{1}{n-1} + \frac{1}{n} \right]; \end{aligned} \quad (38)$$

$$\begin{aligned} \frac{1}{M} [mf_1]_{n-1} &= 1 + \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \int_0^1 dx x^{n-2} \left[ \frac{1+x^2}{1-x_+} + \frac{3}{2} - \frac{3}{2} \right] \\ &= \frac{\alpha_s}{2\pi} C_f \log \frac{Q^2}{\mu^2} \left[ -2 \sum_{j=1}^{n-1} \frac{1}{j} - \frac{1}{n} + \frac{1}{n-1} \right]; \end{aligned} \quad (39)$$

So the moment relation Eq. (36) is satisfied.

To summarize, in this Letter, we investigate the twist three distribution function  $e(x, Q^2)$  for a massive quark at one loop using light-front Hamiltonian perturbation theory in light-front

gauge. By expressing the operator in terms of dynamical fields, we find three terms in the operator ; quark mass term , intrinsic transverse momentum dependent term and an explicit quark-gluon interaction term. The intrinsic transverse momentum dependent part does not give contribution. The equation of motion relation directly relates  $xe(x, Q^2)$  to the twist two unpolarized distribution function  $f_1(x, Q^2)$ . The mass of the quark plays a vital role here. The first moment relation for  $e(x, Q^2)$  is satisfied without a  $\delta(x)$  term in the distribution function. Contribution from the mass term as well as the quark-gluon interaction term diverge as  $x \rightarrow 0$ , however their total contribution does not diverge in this limit. The second moment vanishes in the chiral limit. The logarithmic divergence of  $e(x, Q^2)$  gives the scale dependence.

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